Multifractal stationary random processes

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Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **07 JAN 2005**	2. REPORT TYPE **N/A**	3. DATES COVERED **-**

4. TITLE AND SUBTITLE **Multifractal stationary random processes**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **CMAP, Ecole Polytechnique 91128 Palaiseau Cedex, France**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001750, Wavelets and Multifractal Analysis (WAMA) Workshop held on 19-31 July 2004., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**34**	

Multifractal stationary random processes

- Random cascades (Mandelbrot)
- Intuitive construction
- Theoretical construction
- Some particular examples

Random cascades (Mandelbrot)

Dyadic case

- $l_n = T\lambda^n$, $\lambda = 1/2$
- $I = [0, T[$, $I_0 = [0, T/2[$, $I_{01} = [T/4, T/2[$, ...
- $M_{l_n}(I_{s_0 \ldots s_n}) = W_{s_0 \ldots s_n} M_{l_{n-1}}(I_{s_0 \ldots s_{n-1}})$
- $\{W_{s_0 \ldots s_i}\}$ are i.i.d,
- $\mathbb{E}(W_{s_0 \ldots s_i}) = 1/2$ ($\Rightarrow M_{l_n}$ martingale).
- "Discrete scale invariance" (around 0)

  $\int_0^{t/2} M_{l_n}(dt) =^{law} W \int_0^t M_{l_{n-1}}(dt)$

  $M_{l_n}(dt/2) =^{law} W M_{l_{n-1}}(dt)$

$$M_{l_n}(dt l_n / l_{n-1}) \stackrel{law}{=} W M_{l_{n-1}}(dt)$$

By taking $n \to +\infty$

$$\int_0^{t/2} M(dt) \stackrel{law}{=} W \int_0^t M(dt)$$

$\longrightarrow$ NO : not stationary, no continuous scale invariance

Towards “continuous” cascades

Discrete cascades (dyadic case)

$$M_{l_n}(dt l_n/l_{n-1}) \stackrel{law}{=} W M_{l_{n-1}}(dt)$$

$\Longrightarrow$ Try to represent it as a discretization of an underlying continuous construction.

- $M_{l'}(l'dt) =^{law} W_{l',l}(t) M_l(ldt), \quad l' \leq l \leq T$
- law of $W_{l',l}(t)$ depends only on l'/l (and $i.i.d$).

Why infinitely divisible laws?

- $M_l(dt) =^{law} W_{l,l'}(t) M_{l'}(dt), \quad l \leq l' \leq T$

  $\Rightarrow \ln W_{l',T} = \ln W_{l',l} + \ln W_{l,T}$

  $\Rightarrow \ln W_{l',T} =$ sum of 2 i.i.d variables (for $l = \sqrt{l'T}$)

$\omega_l(t) = \ln W_{l,T}(t) =$ $\sum$ arbitrary number of i.i.d var.

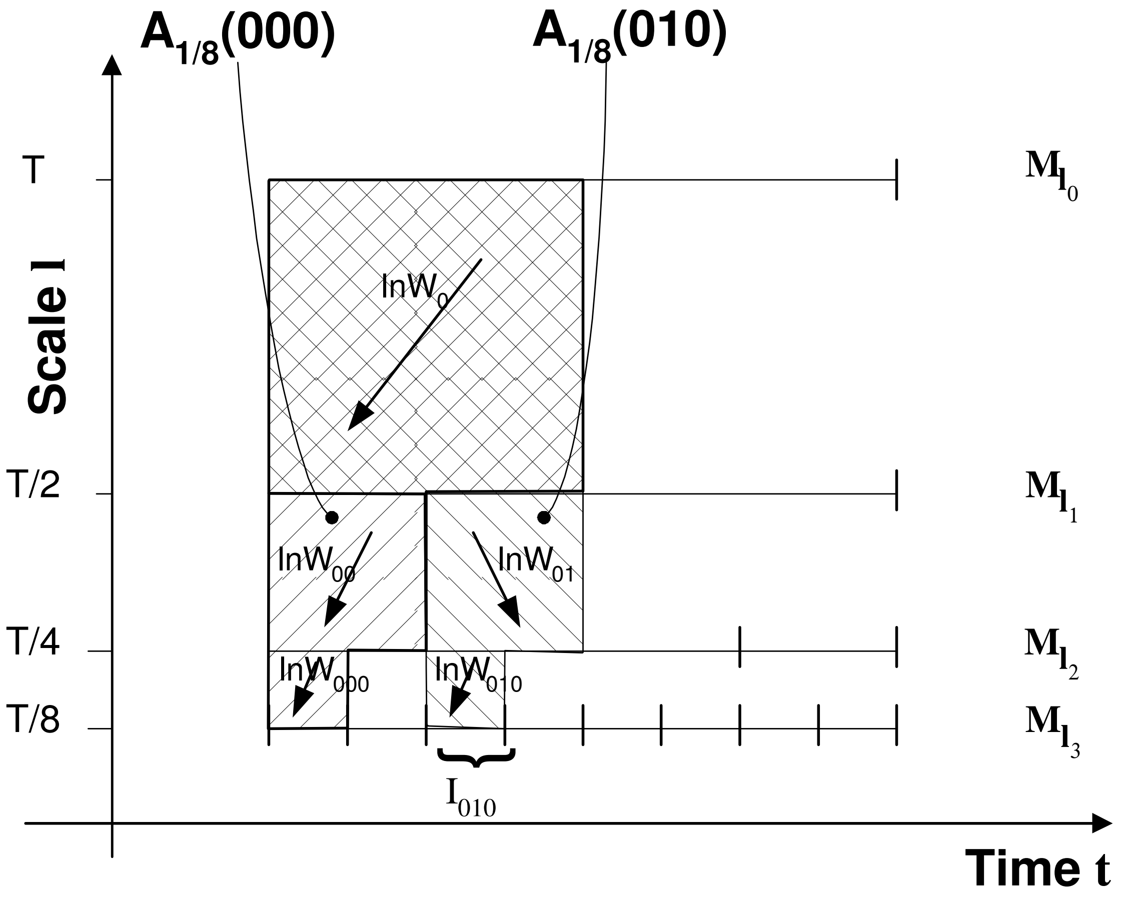
$\Longrightarrow \omega_l(t)$ is infinitely divisible.

- $P(dt, dl)$ stochastic "infinitely divisible noise" on the $\mathcal{S}^+$ half-plane $(t, l) \in \mathbb{R} \times \mathbb{R}^{+*}$ with respect to the measure $\mu(dt, dl)$,

  - $A \subset \mathcal{S}^+$, $P(A)$ infinitely divisible
  - the law of $P(A)$ depends only on $\mu(A)$
  - $P(A)$ and $P(B)$ independent iff $A \cap B = \emptyset$

2d-representation of discrete cascades

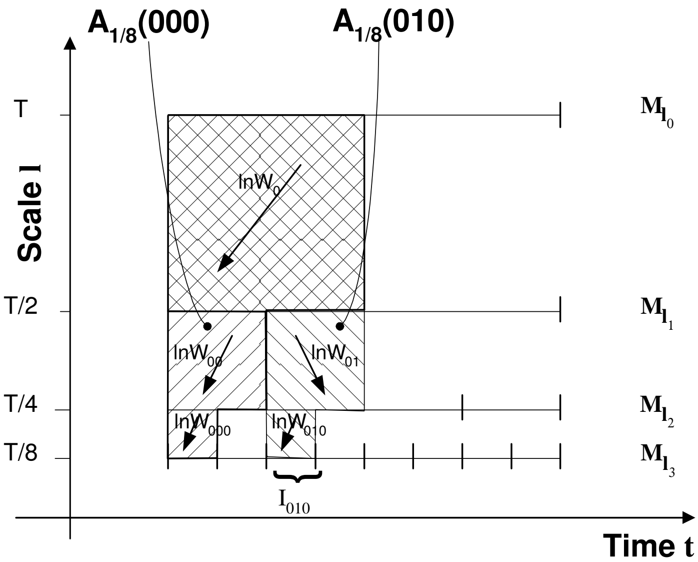
$\omega_{l_n}(s_0 \dots s_n) = \sum_{i=0}^{n} \ln(W_{s_0 \dots s_i})$

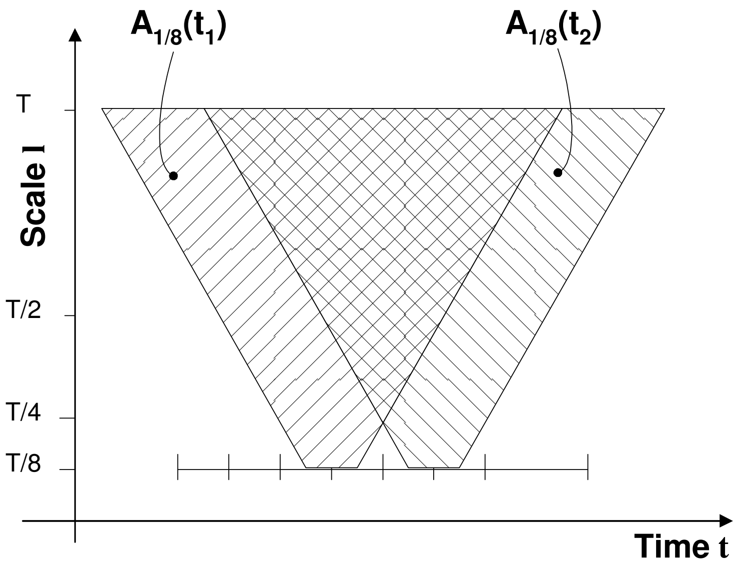


$\omega_{l_n}(s_0 \dots s_n) = P(A_{2^{-n}}(s_0 \dots s_n))$

$\ln W_{s_0 \dots s_i}$ i.d $\Rightarrow$ $\mu(dt, dl) = dt dl / l^2$

Continuous cascades





$$\omega_l(t) = \ln W_{l,T}(t) = P(A_l(t)$$

$$M_l(dt) \stackrel{def}{=} e^{\omega_l(t)} dt$$

$$\Longrightarrow M_{l'}(dt) =^{law} W_{l',l}(t) M_l(dt), \quad l' \le l \le T$$

Infinitely divisible laws

- **Definition**

  X is infinitely divisible iff it is a sum of an arbitrary numbers of i.i.d. variables

- **Examples**

  – Gaussian variable (mean m , variance σ^2)

  $p(x) = e^{-\frac{(x-m)^2}{\sigma^2}} / \sqrt{2\pi\sigma^2}$

  $\mathbb{E}\left(e^{iqX}\right) = e^{imq - q^2\sigma^2/2}$

  $\phi_X(q) = \ln \mathbb{E}\left(e^{iqX}\right) = imq - q^2\sigma^2/2$

  $\phi_X(q)$ is the *cumulant generating function* of X

  – Poisson variable (intensity λ)

  $p(x) = e^{-\lambda}\lambda^k/k!$

  $\phi_X(q) = \lambda(e^{iq} - 1)$

- **Theorem**

A sum of independent infinitely divisible variables is an infinitely divisible variable.

Moreover $\phi_{\alpha X+\beta Y}(q) = \phi_X(\alpha q) + \phi_Y(\beta q)$.

The Levy-Khintchine formula

- Combining gaussian and (an infinite number of) poisson variables :

  $\phi(q) = imq - q^2\sigma^2/2 + \int (e^{iqx} - 1)\tilde{\nu}(dx)$

  $\tilde{\nu}(dx) \simeq$ "intensity measure"

  a priori, total intensity bounded: $\int \tilde{\nu}(dx) < +\infty$

  actually, intensity around 0 can be $+\infty$ $(x^{-2}dx)$

- Levy-Khintchine representation

$$\phi(q) = imq + \int \frac{e^{iqx} - 1 - iq\sin(x)}{x^2}\nu(dx)$$

$\rightarrow$ $\nu(dx)$ is the *Levy measure*

$\rightarrow$ $\int_{-y}^{y} \nu(dx), \int_{-\infty}^{-y} \nu(dx)/x^2, \int_{y}^{+\infty} \nu(dx)/x^2 < +\infty$

– $\nu(dx) = \delta(x)$ $\rightarrow$ gaussian

– $\nu(dx) = \delta(x - x_0)$ $\rightarrow$ poisson

– $\int \nu(dx)/x^2 < +\infty$ $\rightarrow$ compound poisson

– α-stable, gamma, Student, ...

“Infinitely divisible noise”

Definition (*Rosinski, 1989*)
$P(dt, dl)$ is an *independently scattered infinitely divisible random measure* distributed on the half plane $\mathcal{S}^+$ with respect to the (deterministic measure) $\mu(dt, dl)$

- $\{\mathcal{A}_n\}_n$ disjoint sets of $\mathcal{S}^+$, $\{P(\mathcal{A}_n)\}_n$ are i.i.d.

$$P\left(\cup_{n=1}^{\infty}\mathcal{A}_n\right) = \sum_{n=1}^{\infty} P(\mathcal{A}_n)\ , \quad \text{a.s.}$$

- $P(\mathcal{A})$ is infinitely divisible with

$$\mathbb{E}\left(e^{iqP(\mathcal{A})}\right) = e^{\phi(q)\mu(\mathcal{A})},$$

Notation : $\psi(q) = \phi(-iq)$ whenever it is possible and $\psi(q) = +\infty$ otherwise ($\psi(q)$ is convex)

$\rightarrow$ $\psi(q) = +\infty$, if $\mathbb{E}\left(e^{qP(\mathcal{A})}\right) = +\infty$,

$\rightarrow$ $\mathbb{E}\left(e^{qP(\mathcal{A})}\right) = e^{\psi(q)\mu(\mathcal{A})}$, otherwise.

Continuous cascades

$M_l(dt) =^{def} e^{P(A_l(t))} dt$

- *Schmitt F. and Marsan D., 2001*

- basic ideas for construction of $\omega_l(t) = P(\mathcal{A}_l(t))$
- scaling of e^{ω_l} versus l

- *Bacry E., Delour J. and Muzy J.F., 2001*

- 1d representation of the log-normal case (MRW)
- exact power law scaling of the moments
- no proof of convergence

- *Barral J. and Mandelbrot B.B., 2002*

- full 2d compound poisson construction
- non degeneracy, L^p convergence, ...
- multifractal formalism

- *Bacry E., Muzy J.F., 2003*

- full 2d log-inf. divisible construction (MRM, MRW)
- non degeneracy, L^p convergence, ...
- exact/asymptotic scaling of the moments

- *Abry P. , Chainais P., Riedi R. 2003*

- full 2d log-infinitely divisible construction (IDC)
- L^2 convergence
- non power law scaling of the moments

Multifractal Random Measures (MRM)

(Bacry, Muzy, 2003)

- $\mathcal{A}_l(t)$ domain :

  - General definition

  $\mathcal{A}_l(t) = \{(t', l'),\ l' \geq l,\ |t' - t| \leq f(l')/2\}.$

  $\int_l^\infty f(s)/s^2 ds < \infty$ and $f(l) = l$ for $l < L$

  - Definition for "exact scaling"

  $f^{(e)}(l) = l$ for $l \leq T$ and $f^{(e)}(l) = T$ for $l \geq T$

- $\omega_l(t) = P(\mathcal{A}_l(t))$

  - Levy measure $\nu(dx)$

  - $\psi(1) = 0$ ($e^{\omega_l(t)}$ martingale)

  - $\exists \epsilon > 0,\ \psi(1+\epsilon) < +\infty$ ($\mathbb{E}\left(e^{(1+\epsilon)\omega_l(t)}\right) < +\infty$)

- $M_l(dt) = e^{\omega_l(t)} dt$

- $M(dt) = \lim_{l \to 0^+} M(dt)$

Existence of the limit MRM measure

(Bacry, Muzy, 2003)

Theorem 1 *There exists a measure $M(dt)$ such that*

(i) with probability one, $M_l(dt) \to M(dt)$ weakly

(ii) $\forall t \in \mathbb{R}$, $M(\{t\}) = 0$,

(iii) for any bounded set K of $\mathbb{R}$,

- $M(K) < +\infty$
- $\mathbb{E}\left(M(K)\right) \leq |K|$.

Proof

- $\psi(1) = 0 \Longrightarrow \mathbb{E}\left(e^{\omega_l(t)}\right) = 1.$
- $\{M_l(I)\}_l$ is a left continuous positive martingale
- *Kahane J.P., Chi. Ann. of Math.* **8B**, *1-12 (1987)*

$\square$

Continuous scale invariance

(Bacry, Muzy, 2003)

$$M_{l'}(l'dt) = W_{l',l}(t)M_l(ldt), \quad l' \le l \le T.$$

$$\Longrightarrow \quad M_{\lambda l}(\lambda l dt) = W_\lambda(t)M_l(ldt), \quad l' = \lambda l,$$
$$\Longrightarrow \quad M_{\lambda l}(\lambda dt) = W_\lambda(t)M_l(dt),$$
$$\Longrightarrow \quad M_{\lambda l}([0, \lambda t]) = W_\lambda(0)M_l([0, t]).$$

On the one hand

$$M_{\lambda l}([0, \lambda t]) = \int_0^{\lambda t} e^{\omega_{\lambda l}(u)} du = \lambda \int_0^t e^{\omega_{\lambda l}(\lambda u)} du.$$

On the other hand

$$W_\lambda(0)M_l([0, t]) = W_\lambda(0) \int_0^t e^{\omega_l(u)} du.$$

Consequently we want

$$\lambda e^{\omega_{\lambda l}(\lambda t)} \overset{law}{=} W_\lambda \int_0^t e^{\omega_l(u)} du$$

Or,

$$\omega_{\lambda l}(\lambda t) \overset{law}{=} \ln(W_\lambda/\lambda) + \omega_l(t)???$$

Scaling of $\omega_l(t)$

(Bacry, Muzy, 2003)

We want $\{\omega_{\lambda l}(\lambda t)\}_{t \le T} =^{law} \ln(W_\lambda/\lambda) + \{\omega_l(t)\}_{t \le T}$,

Lemma 1 **(Characteristic function of $\omega_l(t)$)**

$$\mathbb{E}\left(e^{\sum_{m=1}^{q} i p_m \omega_l(t_m)}\right) = e^{\sum_{j=1}^{q} \sum_{k=1}^{j} \alpha(j,k)\rho_l(t_k - t_j)},$$

where

- $\rho_l(t) = \mu(\mathcal{A}_l(0) \cap \mathcal{A}_l(t).)$,
- $\sum_{j=1}^{q} \sum_{k=1}^{j} \alpha(j,k) = \varphi\left(\sum_{k=1}^{q} p_k\right)$.

$\square$

The exact scaling domain $f^{(e)}(l)$ is the only domain which satisfies

$$\rho_{\lambda l}(\lambda t) = -\log \lambda + \rho_l(t), \quad l \le T, \ \lambda < 1, \ t < T$$

In this case, $\ln(W_\lambda/\lambda)$ is infinitely divisible (indep. of M) with

$$\mathbb{E}\left(e^{iq \ln(W_\lambda/\lambda)}\right) = \lambda^{-\varphi(q)}.$$

Continuous scale invariance of $M^{(e)}(t)$

(Bacry, Muzy, 2003)

Theorem 2 **(Continuous invariance of $M^{(e)}(t)$)**

$$\{M^{(e)}([0,\lambda t])\}_{t\le T} \stackrel{law}{=} W_\lambda \{M^{(e)}([0,t])\}_{t\le T},$$

where $\ln(W_\lambda/\lambda)$ *is infinitely divisible (indep.of M) with*

$$\mathbb{E}\left(e^{iq\ln(W_\lambda/\lambda)}\right) = \lambda^{-\varphi(q)}.$$

Theorem 3 **(Moment scaling of $M^{(e)}(t)$)**

$$\mathbb{E}\left(M^{(e)}([0,t])^q\right) = \left(\frac{t}{T}\right)^{\zeta(q)} \mathbb{E}\left(M^{(e)}[0,T])^q\right), \quad \forall t \le T.$$

where

$$\zeta(q) = q - \psi(q)$$

Moments, Degeneracy of $M^{(e)}(t)$

(Bacry, Muzy, 2003)

Theorem 4 (Existence of the moments of $M^{(e)}(t)$)
Let $q > 0$ then

(i) $\zeta(q) > 1 \Longrightarrow \mathbb{E}\left(M^{(e)}([0,t])^q\right) < +\infty$ *and* $\sup_l \mathbb{E}\left(M_l^{(e)}([0,t])^q\right) < +\infty$.

(ii) if $M^{(e)} \neq 0$, $\mathbb{E}\left(M^{(e)}([0,t])^q\right) < +\infty \Longrightarrow \zeta(q) \geq 1$.

Proof for (ii)

$$\begin{aligned}
\mathbb{E}\left(M^{(e)}([0,t])^q\right) &= \mathbb{E}\left((M^{(e)}([0,t/2]) + M^{(e)}([t/2,t]))^q\right) \\
&\geq \mathbb{E}\left(M^{(e)}([0,t/2])^q\right) + \mathbb{E}\left(M^{(e)}([t/2,t])^q\right)
\end{aligned}$$

Using scaling of the moments

$$\mathbb{E}\left(M^{(e)}([0,t])^q\right) \geq 2^{1-\zeta(q)}\mathbb{E}\left(M^{(e)}([0,t])^q\right),$$

and consequently $\zeta(q) \geq 1$.

□

Theorem 5 (Non degeneracy of $M^{(e)}(t)$)
(H) $\exists\, \epsilon > 0,\ \zeta(1+\epsilon) > 1$
if (H) holds then $\mathbb{E}\left(M^{(e)}([0,t])\right) = t$.

Summing up results on $M^{(e)}(t)$

(Bacry, Muzy, 2003)

$$
\boxed{
\begin{array}{rl}
\text{(i)} & \psi(1) = 0 \Longrightarrow \text{existence of } M^{(e)} \\
\text{(ii)} & \exists\, \epsilon > 0,\ \zeta(1+\epsilon) > 1 \Longrightarrow \mathbb{E}\left(M^{(e)}([0,t])\right) = t \\
\text{(iii)} & \zeta(q) > 1 \text{ ``} \Longleftrightarrow \text{''}\ \mathbb{E}\left(M^{(e)}([0,t])\right) < +\infty \\
\text{(iv)} & \{M^{(e)}([0,\lambda t])\}_{t \le T} \ =^{law} W_\lambda \{M^{(e)}([0,t])\}_{t \le T}, \\
\text{(v)} & \mathbb{E}\left(M^{(e)}([0,t])^q\right) = \left(\frac{t}{T}\right)^{\zeta(q)} \mathbb{E}\left(M^{(e)}[0,T])^q\right),\ \ \forall t \le T.
\end{array}
}
$$

From $M^{(e)}(t)$ to $M(t)$

(Bacry, Muzy, 2003)

Theorem 6 (Degeneracy, asymptotic scaling and moments of positive orders of $M(dt)$)

(i) $M^{(e)}(dt) =^{a.s.} 0 \Longleftrightarrow M(dt) =^{a.s.} 0,$

Moreover, if $M(dt) \neq 0$, *one has*

(ii) with probability one,

$$\forall t \geq 0, \quad M([0, \lambda t]) \sim X M^{(e)}([0, \lambda t]) \ , \text{ when } \lambda \to 0^+.$$

(iii) $\mathbb{E}\left(M^{(e)}([0,t])^q\right) < +\infty \Longleftrightarrow \mathbb{E}\left(M([0,t])^q\right) < +\infty.$

(iv) $\mathbb{E}\left(M([0,t])^q\right) \sim_{t \to 0^+} \left(\frac{t}{T}\right)^{\zeta(q)} \mathbb{E}\left(M([0,T])^q\right).$

$\square$

Multifractal Random Processes (MRW)

(Bacry, Muzy, 2003)

- **Definition (subordinated process)**

$$X^{(s)}(t) = B(M([0,t])).$$

- **Equivalent definition (stochastic integral)**

$$X(t) = \lim_{l \to 0^+} X_l(t),$$

where

$$X_l(t) = \int_0^t e^{\omega_l(u)/2} dW(u),$$

Main theorem on MRW

(Bacry, Muzy, 2003)

Theorem 7 *Under hypothesis* $\zeta(1+\epsilon) > 1$ *(non degeneracy of* M*),*

(i) $\zeta(q) > 1 \Longrightarrow \mathbb{E}\left(|X(t)|^{2q}\right) < +\infty.$

(ii) $\mathbb{E}\left(|X(t)|^{2q}\right) < +\infty \Longrightarrow \zeta(q) \geq 1.$

(iii) $\{X^{(e)}(t)\}_{t\leq T} \ =^{law} W_\lambda \{X^{(e)}(t)\}_{t\leq T},$

(iv) $\mathbb{E}\left(|X^{(e)}(t)|^{2q}\right) = \left(\frac{t}{T}\right)^{\zeta(q)} \mathbb{E}\left(|X^{(e)}(T)|^{2q}\right), \quad \forall t \leq T.$

(v) $\mathbb{E}\left(|X(t)|^{2q}\right) \ \sim^{t\to 0^+} \ \left(\frac{t}{T}\right)^{\zeta(q)} \mathbb{E}\left(|X(T)|^{2q}\right).$

Numerical simulation of MRW

(Bacry, Muzy, 2003)

- $\{\epsilon[k]\}_{k\in\mathbb{Z}}$ Gaussian white noise, variance 1.
- $l_n = 2^{-n}$.
- $\tilde{X}_{l_n}(t) = \sum_{k=0}^{t/l_n} \sigma\sqrt{l_n} e^{\frac{\omega_{l_n}(kl_n)}{2}} \epsilon[k]$.

Theorem 8 (Convergence) *If* $\zeta(2+\epsilon) > 1$ *then*

$$\lim_{n\to+\infty} \{\tilde{X}_{l_n}(t)\}_t \stackrel{law}{=} \{X(t)\}_t.$$

Proof

$\lim_{n\to+\infty} \tilde{M}_{l_n}(dt) =^{m.s.} M(dt)$

where $\tilde{M}_{l_n}([0,t]) = \sum_{k=0}^{t/l_n} e^{\omega_{l_n}(kl_n)} l_n$

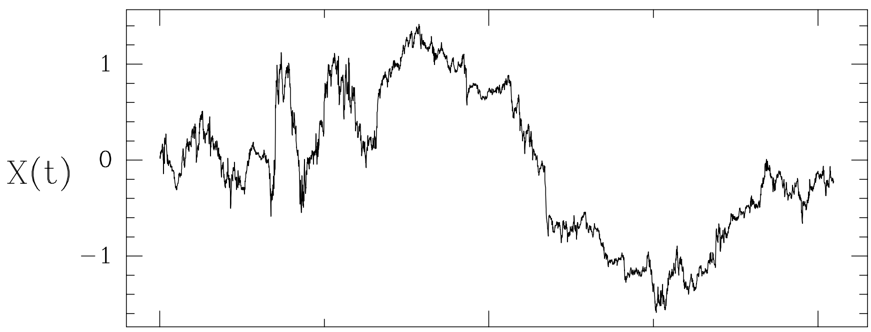
$\square$

$\Longrightarrow X(t)$ is the continuous limit of a random walk with stochastic variance

Log-normal MRW

(Bacry, Muzy, 2003)

- $\nu(dx) = 2\lambda^2\delta(x)$,
- $\psi(q) = qm + \lambda^2 q^2$ with $m = -\lambda^2$.
- $\zeta(q) = q(1+\lambda^2) - \lambda^2 q^2$.



$\sigma^2 = 1$, $T = 512$ and $\lambda^2 = 0.025$.

- **1d construction :** ***Bacry, Delour, Muzy, 2001***

$$X_l(t) = X_l(t-l) + \sigma\sqrt{l}e^{\frac{\omega_l(t)}{2}}\epsilon_l(t), \quad \text{(step-wise process)}$$

$$\mathbb{E}\left(\omega_l\right) = -Var(\omega_l)/2$$

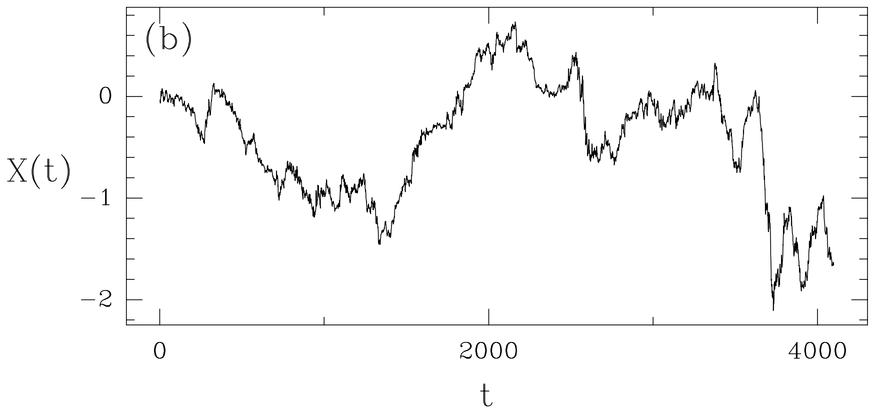
$$Cov(\omega_l(0), \omega_l(t)) = \begin{cases} 4\lambda^2 \ln\left(\frac{T}{(|t|+l)}\right) & \text{for } |t| \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$X(t) = \lim_{l\to 0^+} X_l(t)$$

Log-Poisson MRW

(Bacry, Muzy, 2003 and $\simeq$ Barral, Mandelbrot, 2002)

- $\nu(dx) = \gamma\delta(x - x_0)$, with $\gamma = \lambda x_0^2$, $x_0 = \ln\delta$
- $\zeta(q) = qm + \lambda(1 - \delta^q)$ (m such that $\zeta(1) = 1$)



$\sigma^2 = 1$, $T = 512$, $\lambda = 4$ and $x_0 = 0.05$.

- Log-Poisson compound

  $\int \nu(dx)x^{-2} = C < +\infty$, $p_{\ln W} = \nu(dx)x^{-2}/C$

  Compounded with $\ln W$, intensity C

  $\zeta(q) = qm + C(1 - \mathbb{E}(W^q))$, ($m$ such that $\zeta(1) = 1$)

Other MRW's

(Bacry, Muzy, 2003)

- log-α-stable MRW

  left-sided α-stable density :

$$\nu(dx) = \begin{cases} C|x|^{1-\alpha} & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

  with $C > 0$ and $0 < \alpha < 2$

  $\zeta(q) = qm - \sigma^\alpha |q|^\alpha$

  stable $\Rightarrow$ 1d construction

- log-gamma MRW

  $\nu(dx) = C\gamma^2 x e^{-\gamma x} dx$ for $x \geq 0$

  $\zeta(q) = qm - C\gamma^2 \ln \frac{\gamma}{\gamma - q}$, with $\gamma > 1$

- log-Student, log-Pareto, ...

Log-normal MRW

(Bacry, Muzy, 2001, 2003)

- Continuous limit of a rand. walk with stoch. variance

  $X_l(t) = X_l(t-l) + \sigma^2 \sqrt{l} e^{\frac{\omega_l(t)}{2}} \epsilon_l(t),$ (step-wise)

  $\mathbb{E}\left(\omega_l\right) = -Var(\omega_l)/2$

  $$Cov(\omega_l(0), \omega_l(t)) = \begin{cases} 4\lambda^2 \ln\left(\frac{T}{(|t|+l)}\right) \text{for } |t| \leq T \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

  $X(t) = \lim_{l \to 0^+} X_l(t)$

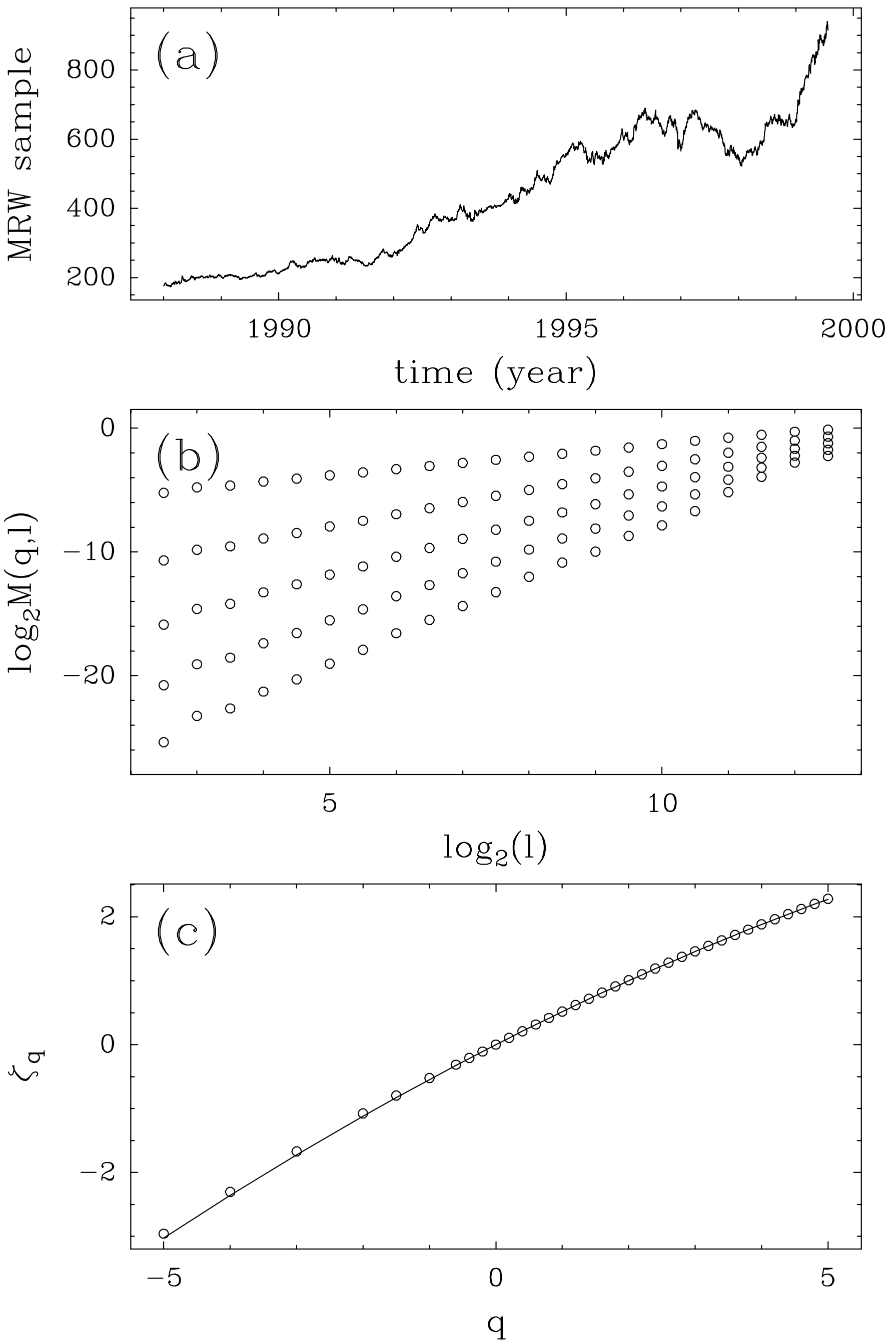
- $\zeta(q) = q - q(q-1)\lambda^2$

- Only 3 parameters :

  - σ^2 : deterministic random walk variance
  - T : correlation length
  - λ^2 : "multifractal coefficient"
    $\lambda = 0 \Rightarrow X(t) = B(t)$

Multifractal analysis of a MRW

(Bacry, Delour, Muzy, 2001)



Moments and law deformation across scales

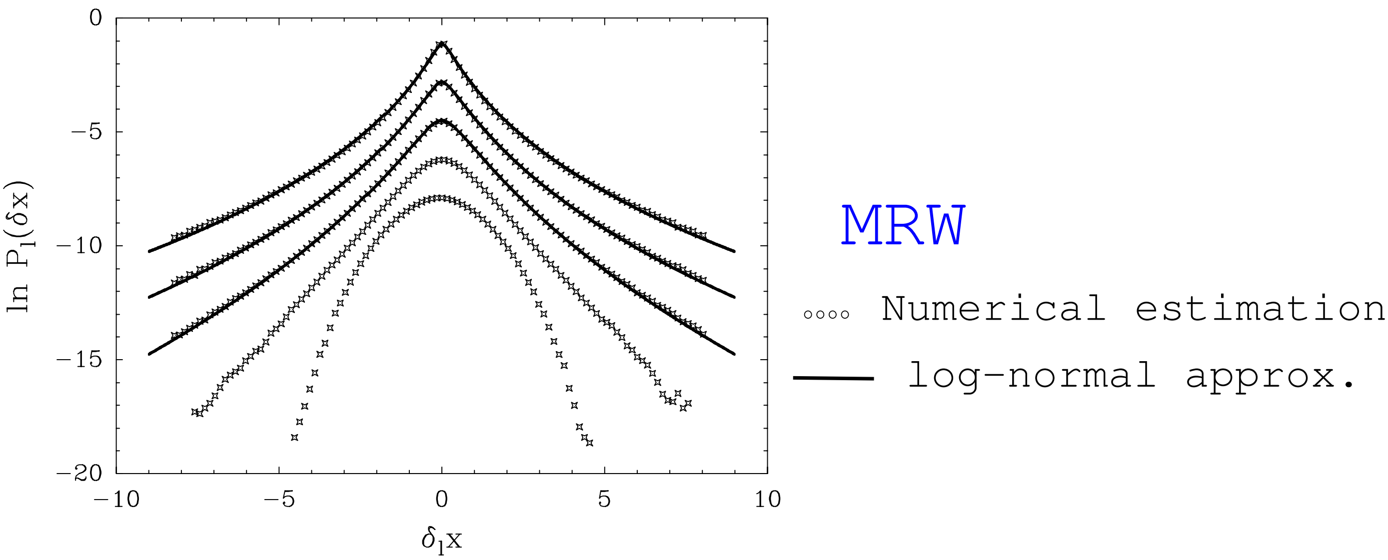
(Bacry, Muzy, 2001, 2003)

- Explicit q-order moment formula

$$\mathbb{E}\left(X(t)^{2q}\right) = C_{2q} T^q \sigma^{2q} (2q-1)!! \left(\frac{t}{T}\right)^{\zeta(q)}, \quad \text{for } t \leq T$$

$$C_q = \prod_{k=0}^{q/2-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2(k+1))}{\Gamma(2-2\lambda^2(q/2+k-1))\Gamma(1-2\lambda^2)}$$

- $q > 1/\lambda^2$ " $\iff$ " $\mathbb{E}\left(X(t)^{2q}\right) = +\infty$



Discretization

(Bacry, Kozhemyak, Muzy, 2004)

- Modeling discrete data with stationary increments
  - Discrete model $X_\Delta[n] = X(n\Delta)$
  - Stationary incr. $\delta X_\Delta[n] = X_\Delta[n+1] - X_\Delta[n]$
- finance, turbulence,...:

  Curvature$(\zeta(q)) = \lambda^2 << 1$
- If $\lambda^2 << 1$, $X_\Delta[n]$ "is close to" rand. walk $R_\Delta[n]$

$$\delta R_\Delta[n] = \epsilon[n] e^{\lambda \Omega_\Delta[n]/2},$$

$\epsilon[n]$: white noise, $\Omega_\Delta[n]$: known Gaussian process.

Theorem 9 **(Meaning of "is close to")**

$$\left\{ \frac{2 \ln M([(n-1)\Delta, n\Delta])}{\lambda} \right\}_n \xrightarrow[\lambda \to 0^+]{law} \{\Omega_\Delta[n]\}_n$$

Moreover,

- $\mathbb{E}\left(\ln |\delta X_\Delta[n]|^q \ldots\right) = \mathbb{E}\left(\ln |\delta R_\Delta[n]|^q \ldots\right)(1 + o(\lambda^{2-\epsilon}))$

and when $\mathbb{E}\left(\delta X_\Delta[n]^q \ldots\right) < +\infty$

- $\mathbb{E}\left(\delta X_\Delta[n]^q \ldots\right) = \mathbb{E}\left(\delta R_\Delta[n]^q \ldots\right)(1 + o(\lambda^{2-\epsilon}))$,

Application to Parameter Estimation

(Bacry, Kozhemyak, Muzy, 2004)

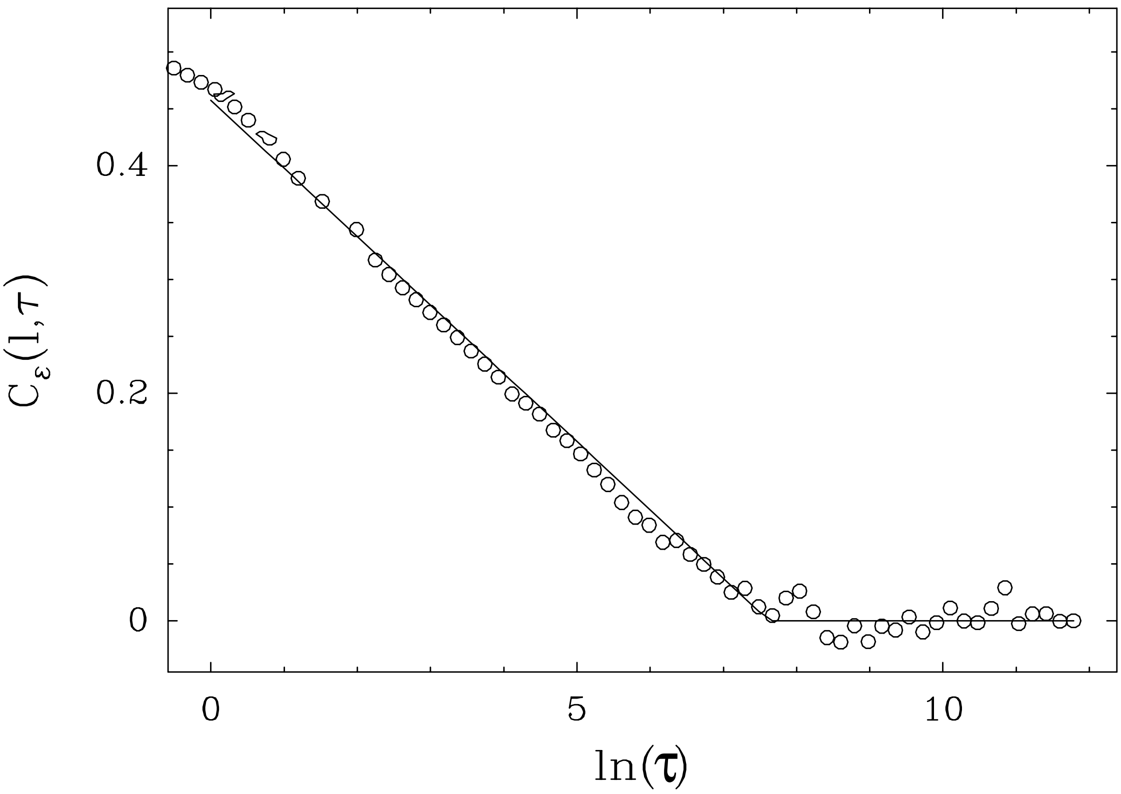
$$
\begin{aligned}
&Cov(\ln|\delta X_\Delta[0]|, \ln|\delta X_\Delta[\tau]|) \\
&\simeq Cov(\ln|\delta R_\Delta[0]|, \ln|\delta R_\Delta[\tau]|) \\
&\simeq Var(\ln|\epsilon|) + \lambda^2 Cov(\Omega_\Delta[0], \Omega_\Delta[\tau])/4 \\
&\simeq -\lambda^2 \ln(\tau/T), \quad \text{for } 1 < \tau < T \\
&\Longrightarrow \text{ independent of } \lambda
\end{aligned}
$$

- Estimation of the multifractal coefficient λ^2
  $\longrightarrow$ linear fit
- Integral scale T
  $\longrightarrow$ decorrelation scale
- Variance σ^2
  $\longrightarrow$ variance of the increments

$\Longrightarrow$ GMM estimation

Correlation function of $\ln|\delta X[n]|$

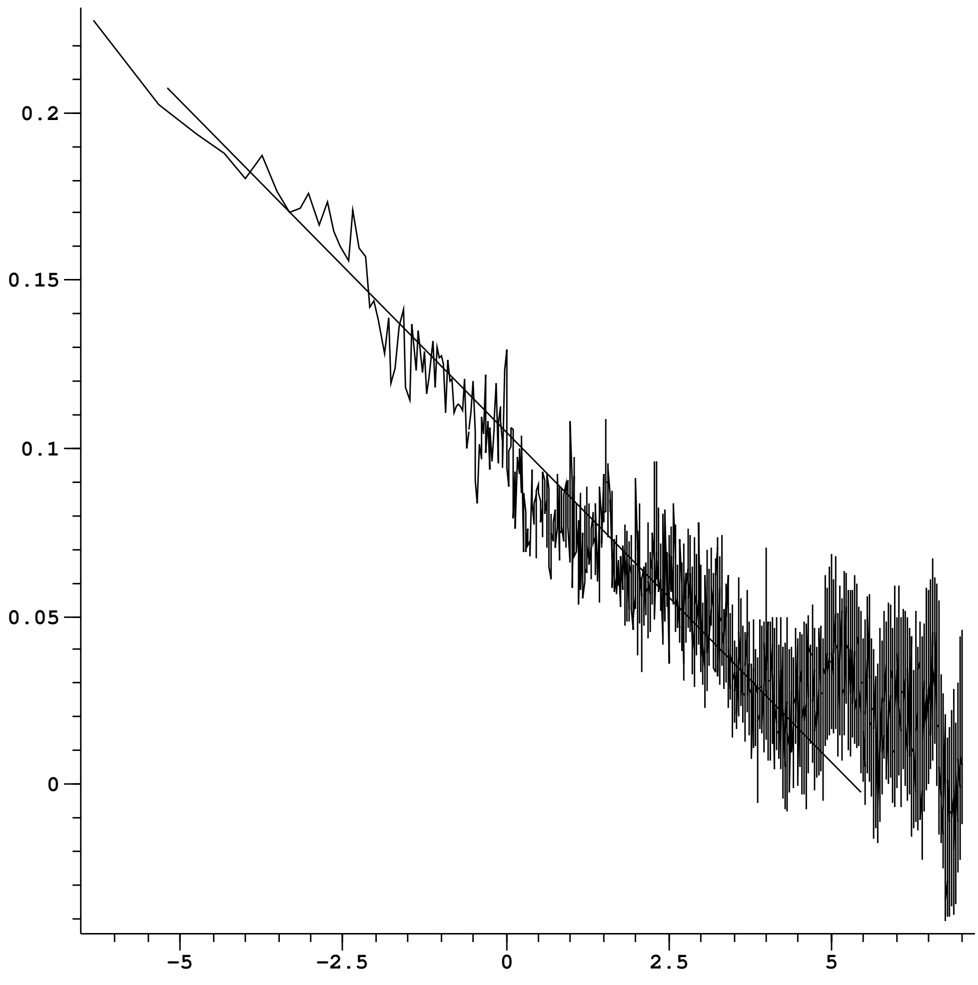
(Bacry, Kozhemyak, Muzy, 2004)



Parameter Estimation

(Bacry, Kozhemyak, Muzy 2004)

S&P 500 intraday data (5mn ticks, 1996-1998)

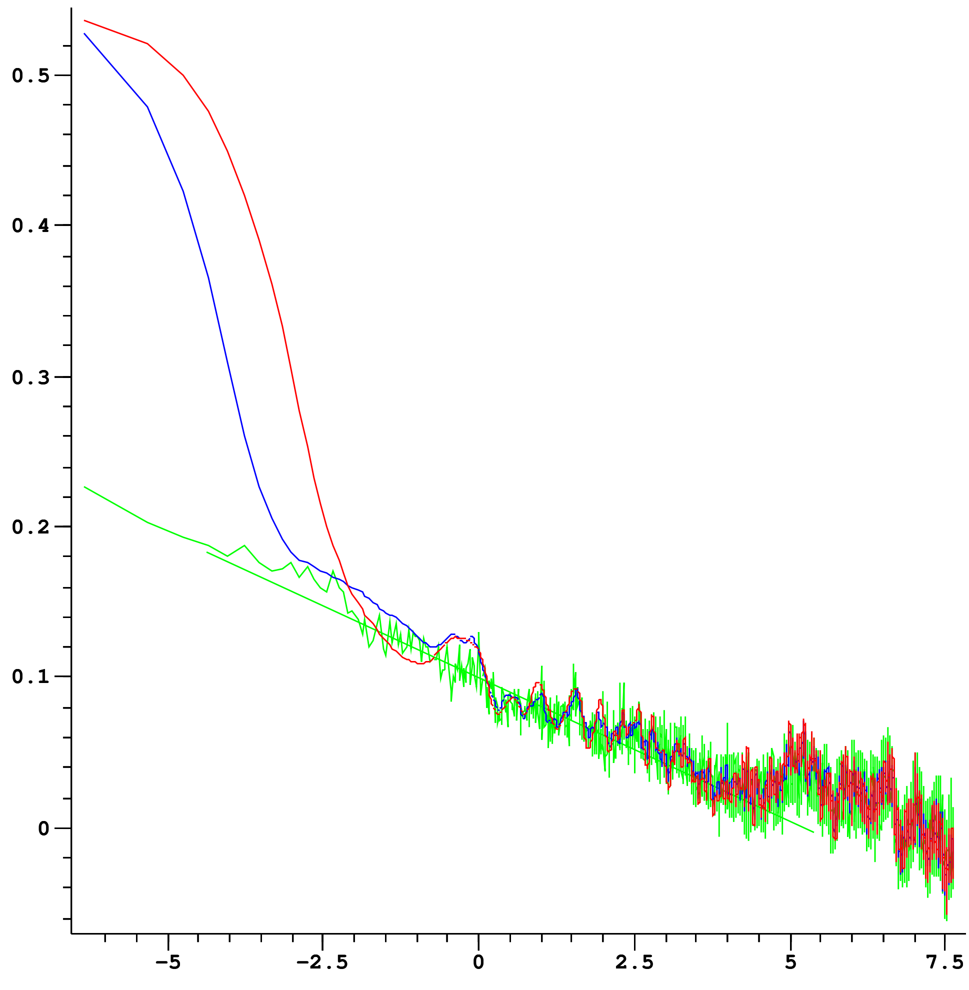


$\lambda^2 \simeq 0.03$, $L \simeq 6$ months

Parameter Estimation
(changing Δ)

(Bacry, Kozhemyak, Muzy 2004)

S&P 500 intraday data (5mn ticks, 1996-1998)



: 5mn returns

: 30mn returns

: 1h returns

Other applications

(Bacry, Kozhemyak, Muzy 2004)

$$\boxed{\delta X_\Delta[n] \simeq \epsilon[n] e^{\lambda \Omega_\Delta[n]/2}}$$

- Variance prediction/estimation (Wiener filtering)
  - Prediction of $\Omega[n]$

    $\hat{\Omega}_\Delta[n] = h_2 * \ln\left(|\delta X_\Delta|\right)[n] \qquad (h_2 \text{ causal})$

    $\longrightarrow$ MLE of $e^{\lambda \Omega_\Delta[n]}$

- Value at Risk prediction/estimation
  - estimation of $v(p)$ such that

$$Prob\left\{|\delta X_\Delta[n]| > v(p) \mid\mid \delta X_\Delta[n-k],\ k > 0\right\} = p$$

$\longrightarrow$ Edgeworth expansion